Non-adaptive Group Testing on Graphs

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Abstract

Alon et al. (2004) [3] introduced the problem of learning a labelled graph from a given family of graphs by non-adaptive algorithms. In this paper, we present a generalization of this problem. Precisely, we consider a graph G and a subgraph H of G and we assume that G contains exactly one defective subgraph isomorphic to H. The goal is to find the defective subgraph by testing whether an induced subgraph contains an edge of the defective subgraph, with minimum number of tests. We give an upper bound for the number of tests to find the defective subgraph by using of the symmetric and high probability variation of Lovász Local Lemma.
Keywords: Group testing on graphs, Non-adaptive algorithm, Combinatorial search, Learning hidden subgraph.

1 Introduction

In the classic *group testing* problem that was first introduced by Dorfman [5] , there is a set of n items which contain at most d defective items. The goal of this problem is to find the defective items with minimum number of tests, which every test can contain some items and each test is positive if contain at least one defective item and otherwise, test is negative. There are two types of algorithms for group testing problem, *sequential* and *non-adaptive*. In the sequential algorithm, the outcome of previous tests can be used in the future tests and in non-adaptive algorithm all tests perform in the same time and by the result of all tests we should find defective items. Group testing has many applications including finding pattern in data, DNA library screening, and so on, for an overview of results and more applications, we refer the reader to [6, 7, 11].

Aigner [1] proposed the problem of *group testing on graphs*. In this problem, we are looking for one defective edge of the given graph G by performing the minimum sequential tests, where each test is an induced subgraph of the graph G and the test is positive if it contain the defective edge.

In this paper, we consider the problem of *non-adaptive group testing on graphs*. Suppose there is a defective subgraph of G isomorphic to a graph H. Each test F is an induced subgraph of G and the outcome of test is positive if F has at least one edge in common with the defective subgraph. This is a generalization of the problem of learning hidden subgraph posed in [2]. More precisely, in learning hidden subgraph problem, the graph G is a complete graph, see [8] for some alternative formulation.

2 Notation

Throughout this paper, we suppose that H is a subgraph of G with k edges and $H_1, H_2, \ldots, H_m$ are all subgraphs of G isomorphic to H. Moreover, we assume that G contains exactly one defective subgraph isomorphic to H. We denote the maximum degree of H by $\Delta = \Delta(H)$. Also, $G[X]$ denotes the subgraph of G induced by $X \cap V(G)$ and for any vertex $v \in G$, $N_H(v)$ stands for the set of neighbours of the vertex v in the graph H. Hereafter, we assume that the subgraph H has no isolated vertices, because in the problem of group testing on graphs, vertices are not defective.

A boolean matrix is said to be *d-disjunct* if for every column C_0 and every choice of d columns $C_1, C_2, \ldots, C_d$ (different from C_0), there is at least one row such that the entry corresponding to C_0 is 1 and the entries corresponding to $C_1, C_2, \ldots, C_d$ are all zeros. This concept was first introduced in [10]. To find the defective subgraph, by Theorem 2.1 of [9], it is sufficient to create a 1-disjunct matrix C such that $C_{ij} = 1$ if the subgraph H_j and the test F_i have at least one edge in common, otherwise $C_{ij} = 0$.

3 Main result

Suppose that $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t \subseteq V(G)$ are t-independent tests such that for any $\mathcal{F}_l$, where $1 \leq l \leq t$, a vertex $v \in V(G)$ is chosen independently with probability p. For simplicity of notation we write F_i as an induced subgraph of G on vertices of $\mathcal{F}_i$.

Let C be a random $t \times m$ matrix such that for any j and l, where $1 \leq j \leq m$ and $1 \leq l \leq t$, if $E(F_l \cap H_j) \neq \varnothing$, then $c_{lj} = 1$; otherwise, $c_{lj} = 0$. For any i, j, l, where $1 \leq i \neq j \leq m$ and $1 \leq l \leq t$, define the event $A^l_{i,j}$ to be the set of all matrices C such that $c_{li} \leq c_{lj}$. Also, define the event $A_{i,j}$ to be the set of all matrices C such that for every l, $1 \leq l \leq t$, we have $c_{li} \leq c_{lj}$. In other word, the event $A^l_{i,j}$ occurs if test F_l cannot distinguish between H_i and H_j. Also, the event $A_{i,j}$ occurs if for every l such that $1 \leq l \leq t$, test F_l cannot distinguish between H_i and H_j.

In the main theorem, we show that the matrix C is a 1-disjunct matrix with positive probability. It means that for every i and j, there are two numbers $1 \leq l, l' \leq t$, such that $E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing$ and $E(H_i \cap F_{l'}) = \varnothing, E(H_j \cap F_{l'}) \neq \varnothing$. Consequently, by Theorem 2.1 of [9] , we can find the defective subgraph with at most t non-adaptive tests.

In the main theorem we use Lovász Local Lemma. In this regard, define the parameter $r_G(H)$, to find the maximum degree of the dependency graph for the events $A_{i,j}$. Define $r_G(H, H_i) = |\{H_j \colon H_j \text{ is isomorphic to } H, V(H_i) \cap V(H_j) \neq \varnothing\}|$ and set $r_G(H) = \max_i r_G(H, H_i)$.

Theorem 1. *Let for the defective subgraph H we have $E(H) = k$, $\Delta(H) = \Delta$. If one of the following conditions holds,*

1. $k > 2\Delta(\Delta - 1) + 1$ *and* $y = \frac{1}{k(\Delta+1)}\left(1 - \frac{1}{\Delta+1}\right)^{2\Delta-1}\left(1 - \sqrt{\frac{1}{k(\Delta+1)}}\right)^{2\Delta-2}$

2. $k \leq 2\Delta(\Delta - 1) + 1$ *and* $y = \frac{1}{2k\Delta}\left(1 - \frac{1}{2\Delta}\right)^{2\Delta-1}\left(1 - \sqrt{\frac{1}{2k\Delta}}\left(1 - \frac{1}{2\Delta}\right)^{\Delta-1}\right)^{2\Delta-2}$

then we can find the defective subgraph H with t non-adaptive tests, where

$$t = 1 + \min\left\{ \lceil \frac{\log(4er_G(H)) + \log m}{\log \frac{1}{1-y}} \rceil, \lceil \frac{2\log m}{\log \frac{1}{1-y}} \rceil \right\}.$$

To prove the main theorem, we need some supportive results.

Lemma 1. *Let H be a graph with n vertices, k edges, and maximum degree Δ. Pick, randomly and independently, each vertex of H with probability p, where $p = \sqrt{\frac{\epsilon}{k}}(1-\frac{\epsilon}{k})^{(\Delta-1)}$. If F is the set of all chosen vertices, then $H[F]$ is an independent set with probability at least $1-\epsilon$.*

To prove this lemma, we need high probability variation of Lovász Local Lemma.

Lemma A. [4] *Let $B_1, B_2, \dots, B_k$ be events in a probability space. Suppose that each event B_i is independent of all the events B_j but at most d. For $1 \le i \le k$ and $0 < \epsilon < 1$, if $Pr(B_i) \le \frac{\epsilon}{k}(1-\frac{\epsilon}{k})^d$, then $Pr\Big(\bigcap_{i=1}^{k} \overline{B_i}\Big) > 1-\epsilon$.*

Proof of Lemma 1. Let $E(H) = \{e_1, e_2, \dots, e_k\}$. For $1 \le i \le k$, we define B_i to be the event that $e_i \in E(H[F])$, so $Pr(B_i) = p^2$. Since vertices are chosen randomly and independently, the event B_i is independent of the event B_j if and only if edges e_i and e_j have no common vertex. So the maximum degree of the dependency graph is at most $2(\Delta-1)$. Since $p^2 \le \frac{\epsilon}{k}\left(1-\frac{\epsilon}{k}\right)^{2(\Delta-1)}$, by Lemma A, $Pr\Big(\bigcap_{i=1}^{k} \overline{B_i}\Big) > 1-\epsilon$. Hence, $H[F]$ is an independent set with probability at least $1-\epsilon$. ■

To find the probability that tests $F_1, F_2, \dots, F_t$ can distinguish between every pair of subgraphs H_i and H_j, we should know the construction of H_i and H_j and their intersection. So we just find the lower bound for this probability. In the next three lemmas we find three lower bounds for this probability in three special cases.

Lemma 2. *Suppose $V(H_i) = V(H_j)$ and $|E(H_i)\backslash E(H_j)| = 1$. If one of the following conditions holds,*

1. $k > 2\Delta(\Delta-1)+1$ *and* $p = \sqrt{\frac{\epsilon}{k}}\left(1 - \frac{\epsilon}{k-2\Delta(\Delta-1)-1}\right)^{\Delta-1}$
2. $k \le 2\Delta(\Delta-1)+1$ *and* $p = \sqrt{\frac{\epsilon}{k}}\,(1-\epsilon)^{\Delta-1}$

then $Pr\big(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\big) \ge p^2(1-p)^{2(\Delta-1)}(1-\epsilon)$.

Proof. Let $e = \{u, v\} \in E(H_i) \setminus E(H_j)$. If $H_j \cap F_l$ is an independent set of H_j and $u, v \in \mathcal{F}_l$, then $E(H_i \cap F_l) \neq \varnothing$ and $E(H_j \cap F_l) = \varnothing$. Suppose that H' is an induced subgraph of H_j, where $V(H') = V(H_j) \setminus \Big(u \cup v \cup N(u) \cup N(v)\Big)$ and $|E(H')| = k'$. So $k' \ge k - 2\Delta(\Delta-1) - 1$ and $|V(H')| \ge |V(H)| - 2\Delta$.

It is easily seen that $u, v \in \mathcal{F}_l$ and $H_j[F_l]$ is an independent set if and only if $u, v \in \mathcal{F}_l$, $N_{H_j}(u) \cap \mathcal{F}_l = \varnothing$, $N_{H_j}(v) \cap \mathcal{F}_l = \varnothing$, and $H'[F_l]$ is an independent set.

One can see that $Pr(u, v \in \mathcal{F}_l) = p^2$. Since $|N_{H_j}(u) \cup N_{H_j}(v) \setminus \{u, v\}| \leq 2(\Delta - 1)$, the probability that $\mathcal{F}_l \cap \Big(N_{H_j}(u) \cup N_{H_j}(v) \setminus \{u, v\}\Big) = \varnothing$ is at least $(1-p)^{2\Delta-2}$.

1. Suppose $k > 2\Delta(\Delta-1)+1$ and $p = \sqrt{\frac{\epsilon}{k}}\left(1 - \frac{\epsilon}{k - 2\Delta(\Delta-1)-1}\right)^{\Delta-1}$. One can check $p^2 \leq \frac{\epsilon}{k'}\left(1-\frac{\epsilon}{k'}\right)^{2\Delta-2}$, and by Lemma 1, the induced subgraph on $\mathcal{F}_l \cap V(H')$ is independent set with probability at least $1-\epsilon$; in other words, $Pr\big(E(F_l \cap H') = \varnothing\big) \geq 1-\epsilon$. Hence,

$$Pr\Big(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\Big) \geq p^2(1-p)^{2\Delta-2}(1-\epsilon).$$

2. Now suppose $k \leq 2\Delta(\Delta-1)+1$ and $p = \sqrt{\frac{\epsilon}{k}}\,(1-\epsilon)^{\Delta-1}$. If $k' = 0$, then $Pr\big(E(F_l \cap H') = \varnothing\big) = 1$. Else if $k' \geq 1$, One can check $p^2 \leq \frac{\epsilon}{k'}\left(1-\frac{\epsilon}{k'}\right)^{2\Delta-2}$, and by Lemma 1, the induced subgraph on $\mathcal{F}_l \cap V(H')$ is an independent set with probability at least $1-\epsilon$. In other words, $Pr\big(E(F_l \cap H') = \varnothing\big) \geq 1-\epsilon$. Hence, $Pr\big(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\big) \geq p^2(1-p)^{2\Delta-2}(1-\epsilon)$. ■

In the previous lemma two subgraphs H_i and H_j are very similar. By analysis of next two cases, we show that in this case the probability of distinguishing between H_i and H_j is minimum.

Lemma 3. *Suppose $|V(H_i) \setminus V(H_j)| = 1$. If either*
$k > 2\Delta(\Delta-1)+1$ *and* $p = \sqrt{\frac{\epsilon}{k}}\left(1 - \frac{\epsilon}{k-2\Delta(\Delta-1)-1}\right)^{\Delta-1}$, *or*
$k \leq 2\Delta(\Delta-1)+1$ *and* $p = \sqrt{\frac{\epsilon}{k}}\,(1-\epsilon)^{\Delta-1}$, *then*

$$Pr\Big(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\Big) \geq p^2(1-p)^{\Delta}(1-\epsilon).$$

Proof. Since H has no isolated vertices, one can see that there exits at least one edge $e = \{u, v\} \in E(H_i) \setminus E(H_j)$. Let $v \in V(H_i) \cap V(H_j)$ and $u \in V(H_i) \setminus V(H_j)$. Suppose that H' is an induced subgraph of H_j, where $V(H') = V(H_j) \setminus (v \cup N(v))$ and $|E(H')| = k'$. So $k^{'} \geq k - \Delta^2$ and $|V(H')| \geq |V(H)| - \Delta - 1$.

Similar to the proof of Lemma 2, if $u, v \in \mathcal{F}_l$, $N_{H_j}(v) \cap \mathcal{F}_l = \varnothing$, and $H'[F_l]$ is an independent set, then $E(H_i \cap F_l) \neq \varnothing$ and $E(H_j \cap F_l) = \varnothing$.

Since $|N_{H_j}(v)| \leq \Delta$, the probability that $N_{H_j}(v) \cap \mathcal{F}_l = \varnothing$ and $u, v \in \mathcal{F}_l$ is at least $p^2(1-p)^{\Delta}$.

If $k > 2\Delta(\Delta-1)+1$ and $p = \sqrt{\frac{\epsilon}{k}}\left(1 - \frac{\epsilon}{k-2\Delta(\Delta-1)-1}\right)^{\Delta-1}$, then

$$p^2 \leq \frac{\epsilon}{k}\left(1 - \frac{\epsilon}{k-\Delta^2}\right)^{2\Delta-2} \leq \frac{\epsilon}{k'}\left(1-\frac{\epsilon}{k'}\right)^{2\Delta-2}.$$

Hence, by Lemma 1, $Pr\left(E(F_l \cap H') = \varnothing\right) \geq 1-\epsilon$, which implies

$$Pr\Big(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\Big) \geq p^2(1-p)^{\Delta}(1-\epsilon),$$

Now suppose $k \leq 2\Delta(\Delta-1)+1$ and $p = \sqrt{\frac{\epsilon}{k}}\,(1-\epsilon)^{\Delta-1}$. If $k' = 0$, then $Pr\big(E(F_l \cap H') = \varnothing\big) = 1$. Else if $k' \geq 1$, One can check $p^2 \leq \frac{\epsilon}{k'}\left(1-\frac{\epsilon}{k'}\right)^{2\Delta-2}$, and by Lemma 1, $Pr\left(E(F_l \cap H') = \varnothing\right) \geq 1-\epsilon$.

Hence, $Pr\Big(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\Big) \geq p^2(1-p)^{\Delta}(1-\epsilon)$. ■

Lemma 4. *If the induced subgraph on $V(H_i) - V(H_j)$ has at least one edge, then*

$$Pr\Big(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\Big) \geq p^2(1-\epsilon),$$

where $p = \sqrt{\frac{\epsilon}{k}}\left(1 - \frac{\epsilon}{k}\right)^{2\Delta-2}$.

Proof. Let $e = (u,v) \in E(H_i) \setminus E(H_j)$. If $H_j[F_l]$ is an independent set and $u, v \in \mathcal{F}_l$, then $E(H_i \cap F_l) \neq \varnothing$ and $E(H_j \cap F_l) = \varnothing$. Since $Pr(u, v \in \mathcal{F}_l) = p^2$ and $p^2 \leq \frac{\epsilon}{k}\left(1 - \frac{\epsilon}{k}\right)^{2\Delta-2}$, by Lemma 1, $Pr\left(E(H_j \cap F_l) = \varnothing\right) \geq 1 - \epsilon$; and consequently, $Pr\big(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\big) \geq p^2(1-\epsilon)$. ■

In the three previous lemmas, we find three lower bounds for the probability of distinguishing between H_i and H_j. By the use of next lemma we generalize these three special case to general case. The proof of the following lemma is straightforward.

Lemma 5. *If $E(H_i) \cap E(H_j) = \{f_1, f_2, \ldots, f_r\}$ and $E(H_i) \setminus E(H_j) = \{e_1, e_2, \ldots, e_{k-r}\}$, then for every $1 \leq q \leq k - r$ and $1 \leq l \leq t$,*

$$Pr\Big(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\Big) \geq Pr\Big(e_q \in E(F_l \cap H_i), E(H_j \cap F_l) = \varnothing\Big).$$

In the next theorem, we show that the probability of distinguishing between H_i and H_j has the minimum value, whenever $V(H_i) = V(H_j)$ and $|E(H_i) \setminus E(H_j)| = 1$.

Theorem 2. *Let $|E(H)| = k$ and $\Delta(H) = \Delta$. If either*

1. $k > 2\Delta(\Delta-1) + 1$, $\epsilon = \frac{1}{\Delta+1}$ *and* $p = \sqrt{\frac{\epsilon}{k}}\left(1 - \frac{\epsilon}{k - 2\Delta(\Delta-1) - 1}\right)^{\Delta-1}$, *or*
2. $k \leq 2\Delta(\Delta-1) + 1$, $\epsilon = \frac{1}{2\Delta}$ *and* $p = \sqrt{\frac{\epsilon}{k}}(1-\epsilon)^{\Delta-1}$,

then for every $1 \leq i \neq j \leq m$ and $1 \leq l \leq t$, we have

$$Pr\left(\overline{A_{i,j}^l}\right) \geq p^2(1-p)^{2\Delta-2}(1-\epsilon). \tag{1}$$

Proof. Let $E(H_i) \cap E(H_j) = \{f_1, f_2, \ldots, f_r\}$ and $E(H_i) \setminus E(H_j) = \{e_1, e_2, \ldots, e_{k-r}\}$. The event $\overline{A_{i,j}^l}$ occurs if $E(H_i \cap F_l) \neq \varnothing$ and $E(H_j \cap F_l) = \varnothing$. By Lemma 5, for every $1 \leq q \leq k - r$,

$$Pr\Big(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\Big) \geq Pr\Big(e_q \in E(F_l \cap H_i), E(H_j \cap F_l) = \varnothing\Big),$$

so we need to consider the following three cases,

case 1: $V(H_i) = V(H_j)$, $|E(H_i) \setminus E(H_j)| = 1$.

By Lemma 2, it is clear $Pr\left(\overline{A_{i,j}^l}\right) \geq p^2(1-p)^{2(\Delta-1)}(1-\epsilon)$.

case 2: $|V(H_i) \setminus V(H_j)| = 1$.

By Lemma 3, we have $Pr(\overline{A_{i,j}^l}) \geq p^2(1-p)^{\Delta}(1-\epsilon) \geq p^2(1-p)^{2(\Delta-1)}(1-\epsilon)$.

case 3: The induced subgraph on $V(H_i) - V(H_j)$ has at least one edge.

One can check $p^2 \leq \frac{\epsilon}{k}\left(1-\frac{\epsilon}{k}\right)^{2\Delta-2}$, hence, by Lemma 4, $Pr\left(\overline{A^l_{i,j}}\right) \geq p^2(1-\epsilon)$.

Consequently, $Pr\left(\overline{A^l_{i,j}}\right) \geq p^2(1-p)^{2\Delta-2}(1-\epsilon)$.

So for every $1 \leq i \neq j \leq m$ and $1 \leq l \leq t$, $Pr\left(\overline{A^l_{i,j}}\right) \geq p^2(1-p)^{2\Delta-2}(1-\epsilon)$. ∎

To prove the main theorem, at first we should find an upper bound for the probability of occurring the bad event $A_{i,j}$ for every $1 \leq i \neq j \leq m$.

Theorem 3. *Let $|E(H)| = k$ and $\Delta(H) = \Delta$. If either*

1. $k > 2\Delta(\Delta-1)+1$ *and* $y = \frac{1}{k(\Delta+1)}\left(1-\frac{1}{\Delta+1}\right)^{2\Delta-1}\left(1-\sqrt{\frac{1}{k(\Delta+1)}}\right)^{2\Delta-2}$, *or*

2. $k \leq 2\Delta(\Delta-1)+1$ *and* $y = \frac{1}{2k\Delta}\left(1-\frac{1}{2\Delta}\right)^{2\Delta-1}\left(1-\sqrt{\frac{1}{2k\Delta}}\left(1-\frac{1}{2\Delta}\right)^{\Delta-1}\right)^{2\Delta-2}$,

then for every $1 \leq i \neq j \leq m$

$$Pr(A_{i,j}) \leq (1-y)^t. \tag{2}$$

Proof. Since $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t \subset V(G)$ are chosen randomly and independently, for every $1 \leq l \neq l' \leq t$, $A^l_{i,j}$ and $A^{l'}_{i,j}$ are two independent events. So

$$Pr(A_{i,j}) = \left(Pr(A^l_{i,j})\right)^t.$$

Note that $Pr\left(\overline{A^l_{i,j}}\right) = Pr\left(E(H_i \cap F_l) \neq \varnothing, E(H_j \cap F_l) = \varnothing\right)$.

1. If $k > 2\Delta(\Delta-1)+1$, $\epsilon = \frac{1}{\Delta+1}$ and $p = \sqrt{\frac{\epsilon}{k}}\left(1-\frac{\epsilon}{k-2\Delta(\Delta-1)-1}\right)^{\Delta-1}$, then by Theorem 2, $Pr\left(\overline{A^l_{i,j}}\right) \geq y'y''$, where

$$y' = \frac{\Delta}{k(\Delta+1)^2}\left(1-\frac{1}{(\Delta+1)(k-2\Delta(\Delta-1)-1)}\right)^{2\Delta-2},$$

$$y'' = \left(1-\sqrt{\frac{1}{k(\Delta+1)}}\left(1-\frac{1}{(\Delta+1)(k-2\Delta(\Delta-1)-1)}\right)^{\Delta-1}\right)^{2\Delta-2}.$$

one can easily seen that
$y'y'' > y = \frac{1}{k(\Delta+1)}\left(1-\frac{1}{\Delta+1}\right)^{2\Delta-1}\left(1-\sqrt{\frac{1}{k(\Delta+1)}}\right)^{2\Delta-2}$. Hence, $Pr(\overline{A^l_{i,j}}) > y$, and consequently, $Pr(A_{i,j}) < (1-y)^t$.

2. Now suppose $k \leq 2\Delta(\Delta-1)+1$, $\epsilon = \frac{1}{2\Delta}$, and $p = \sqrt{\frac{\epsilon}{k}}(1-\epsilon)^{\Delta-1}$. By Theorem 2, $Pr\left(\overline{A^l_{i,j}}\right) \geq y$, where

$$y = \frac{1}{2k\Delta}\left(1-\frac{1}{2\Delta}\right)^{2\Delta-1}\left(1-\sqrt{\frac{1}{2k\Delta}}\left(1-\frac{1}{2\Delta}\right)^{\Delta-1}\right)^{2\Delta-2}.$$

Hence, $Pr(A_{i,j}) < (1-y)^t$. ∎

Now, we are ready to prove the main theorem.

Proof of Theorem 1. If either

1. $k > 2\Delta(\Delta-1)+1$ and $y = \frac{1}{k(\Delta+1)}\left(1-\frac{1}{\Delta+1}\right)^{2\Delta-1}\left(1-\sqrt{\frac{1}{k(\Delta+1)}}\right)^{2\Delta-2}$, or

2. $k \leq 2\Delta(\Delta-1)+1$ and $y = \frac{1}{2k\Delta}\left(1-\frac{1}{2\Delta}\right)^{2\Delta-1}\left(1-\sqrt{\frac{1}{2k\Delta}}\left(1-\frac{1}{2\Delta}\right)^{\Delta-1}\right)^{2\Delta-2}$,

then by Theorem 3, for every $1 \leq i \neq j \leq m$, $Pr(A_{i,j}) \leq (1-y)^t$. Since the number of events $A_{i,j}$ is m^2, the probability that at least one of $A_{i,j}$'s occurs is at most $m^2(1-y)^t$. If $t > \frac{2\log m}{\log\frac{1}{1-y}}$, then $m^2(1-y)^t < 1$.

Now we prove that if $t > \frac{\log(4er_G(H))+\log m}{\log\frac{1}{1-y}}$, then by Lovász Local Lemma, with positive probability no event $A_{i,j}$ occurs. Each event $A_{i,j}$ is independent of $A_{i',j'}$ if and only if $\Big(V(H_i)\cup V(H_j)\Big)\cap\Big(V(H_{i'})\cup V(H_{j'})\Big)=\varnothing$. So the maximum degree of the dependency graph is at most $4r_G(H)(m-1)$. Hence, if $t > \frac{\log(4er_G(H))+\log m}{\log\frac{1}{1-y}}$, then $e(1-y)^t(4r_G(H)(m-1)+1) < 1$. So by Lovász Local Lemma $Pr\Big(\bigcap\limits_{i,j}\overline{A_{i,j}}\Big) > 0$.

Therefore, if $t = 1+\min\left\{\lceil\frac{\log(4er_G(H))+\log m}{\log\frac{1}{1-y}}\rceil, \lceil\frac{2\log m}{\log\frac{1}{1-y}}\rceil\right\}$, then there is a $t\times m$ disjunct matrix. ■

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